Examples of Proving Asymptotic Bounds on Computational Complexity Functions

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Prior to reading this handout, students should read a tutorial paper entitled “Writing a Math Phase Two Paper” [1], ICS141 Lecture Notes on Predicate Logic (Part 3 and Part 4),
http://pearl.ics.hawaii.edu/ sugihara/courses/ics141s15/notes/PredLogic3.html
http://pearl.ics.hawaii.edu/ sugihara/courses/ics141s15/notes/PredLogic4.html
and ICS141 Lecture Notes on Predicate Logic on Algorithm (Part 1).
http://pearl.ics.hawaii.edu/ sugihara/courses/ics141s15/notes/Algorithms1.html

1. Review of Asymptotic Notations

There are four asymptotic notations covered in ICS141 [2], [3].
(a) **Upper Bound:** “Big-Oh” notation $f(n) = O(g(n))$ or $f(n) \in O(g(n))$
(b) **Lower Bound:** “Big-Omega” notation $f(n) = \Omega(g(n))$ or $f(n) \in \Omega(g(n))$
(c) **Tight Bound:** “Theta” notation $f(n) = \Theta(g(n))$ or $f(n) \in \Theta(g(n))$
(d) **Dominated Upper Bound:** “little-oh” notation $f(n) = o(g(n))$ or $f(n) \in o(g(n))$

In the analysis of computational complexities of algorithms, we always attempt to derive a tight bound in the $\Theta$ notation. If it is difficult to derive a lower bound, then we compromise with an upper bound rather than a tight bound. A few examples of proving a tight bound are given below.

**Definition 1.** $f(n) = O(g(n))$ iff $\exists c \in \mathbb{R} \ \exists N \in \mathbb{Z}^+ \ [ \forall n \in \mathbb{Z}^+ \ [ n \geq N \rightarrow f(n) \leq c|g(n)| ] ]$

**Definition 2.** $f(n) = \Omega(g(n))$ iff $\exists c \in \mathbb{R} \ \exists N \in \mathbb{Z}^+ \ [ \forall n \in \mathbb{Z}^+ \ [ n \geq N \rightarrow c|g(n)| \leq f(n) ] ]$

**Definition 3.** $f(n) = \Theta(g(n))$ iff $f(n) = O(g(n)) \land f(n) = \Omega(g(n))$

By Definition 3, one way to prove a tight bound is to prove two assertions (i.e., an upper bound and a lower bound) separately. To prove each of the two bounds requires to give concrete values of the two
constants $c$ and $N$ satisfying the corresponding inequality.

Definition 3 can be rephrased as follows.

**Definition 4.**

\[ f(n) = \Theta(g(n)) \iff \exists c_1 \in \mathbb{R} \exists c_2 \in \mathbb{R} \exists N \in \mathbb{Z}^+ \ [ \forall n \in \mathbb{Z}^+ \ [ n \geq N \rightarrow c_1 |g(n)| \leq f(n) \leq c_2 |g(n)| ] ] \]

This definition suggests another way to prove a tight bound that is to give concrete values of the three constants $c_1$, $c_2$ and $N$ satisfying the two inequalities in Definition 4.

**Ex.1 Prove or disprove** \(2x^2 + x - 7 = \Theta(x^2)\).

(Section 3.2 Ex.24(b) in p.191 of the Rosen’s textbook [2] for ICS141)

**Theorem 1.** \(2x^2 + x - 7 = \Theta(x^2)\)

*Proof:* Let $c_1 = 2$, $c_2 = 3$ and $N = 7$.

\[ c_1 |x^2| = 2x^2 \leq 2x^2 + x - 7 \quad \text{for all } x \geq N \]

since $7 \leq x$ for all $x \geq N$

\[ 2x^2 + x - 7 < 2x^2 + x < 2x^2 + x^2 = 3x^2 = c_2 x^2 \quad \text{for all } x \geq N \]

since $x < x^2$ for all $x \geq N$. \[\blacksquare\]

**Ex.2 Prove or disprove** \(x \log x = \Theta(x^2)\).

**Theorem 2.** \(x \log x = \Theta(x^2)\) is false.

*Proof:* We disprove the tight bound by proving $x \log x = o(x^2)$.

\[ \lim_{x \to \infty} \frac{x \log x}{x^2} = \lim_{x \to \infty} \frac{\log x}{x} \]

\[ = \lim_{x \to \infty} \frac{d}{dx} \log x \]

\[ = \lim_{x \to \infty} \frac{d}{dx} \frac{d}{dx} x \quad \text{by L’Hôpital’s Rule} \]

\[ = \lim_{x \to \infty} \frac{1}{x} \quad \text{since } \frac{d}{dx} \log x = \frac{1}{x} \]

\[ = \lim_{x \to \infty} \frac{1}{x} \]

\[ = 0 \]
Thus, by the definition of the little-o notation, \( x \log x = o(x^2) \) holds. Since \( f(n) = \Omega(g(n)) \) and \( f(n) = o(g(n)) \) cannot hold simultaneously, the tight bound \( x \log x = \Theta(x^2) \) does not hold.

**Ex.3 Prove or disprove** \( \sum_{i=1}^{n} i(i - 1) = \Theta(n^3) \).

**Theorem 3.** \( \sum_{i=1}^{n} i(i - 1) = \Theta(n^3) \)

*Proof:* We first simplify the summation.

\[
\sum_{i=1}^{n} i(i - 1) = \sum_{i=1}^{n} (i^2 - i) = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i = \frac{n(n + 1)(2n + 1)}{6} - \frac{n(n + 1)}{2}
\]

by closed formulas for the finite series in Section 2.4 Table 2 (p.157) of the Rosen’s Textbook [4]

\[
= 2n^3 + 3n^2 + n - 3(n^2 + n)
\]

\[
= 2n^3 - 2n
\]

\[
= \frac{n^3 - n}{3}
\]

Next, we show an upper bound \( O(n^3) \) as follows.

\[
\frac{n^3 - n}{3} \leq \frac{n^3}{3} = \frac{1}{3}n^3 = c|n^3| \text{ for all } n \geq N \text{ where } c = \frac{1}{3} \text{ and } N = 1.
\]

Next, we show a lower bound \( \Omega(n^3) \) as follows.

\[
\frac{n^3 - n}{3} = \frac{n^3}{3} - \frac{n}{3} = \frac{n^3}{6} + \frac{(n^3 - 2n)}{6} > \frac{n^3}{6} \text{ for all } n \geq 2 \text{ since } n^3 > 2n \text{ for all } n \geq 2
\]

Thus, \( \frac{n^3 - n}{3} \geq c \cdot |n^3| \) holds for all \( n \geq N \), where \( c = \frac{1}{6} \) and \( N = 2 \).

Therefore, the tight bound \( \sum_{i=1}^{n} i(i - 1) = \Theta(n^3) \) holds.
Ex. 4 Prove or disprove $\ln(x^2 + 1) = \Theta(\lg x)$.

Theorem 4. $\ln(x^2 + 1) = \Theta(\lg x)$

Proof: We first convert the natural logarithm $\ln$ to $\lg$, i.e., the base of logarithm is converted from $e$ to 2 by using the following fact (See Appendix 2 Theorem 3 in p.A-8 [5]). $\log_a x = \frac{\log_b x}{\log_b a}$

$\ln(x^2 + 1) = \frac{\lg(x^2 + 1)}{\lg e}$ where $e$ is the base of the natural logarithm called the Napier constant

Next, we prove an upper bound.

$$\frac{\lg(x^2 + 1)}{\lg e} \leq \frac{\lg(x^2 + x^2)}{\lg e} \quad \text{for all } x \geq 1 \quad \text{since } 1 \leq x^2 \text{ for all } x \geq 1$$

$$= \frac{\lg(2x^2)}{\lg e} \quad \text{for all } x \geq 1$$

$$= \frac{\lg 2 + \lg x^2}{\lg e} \quad \text{for all } x \geq 1 \quad \text{since } \log_b xy = \log_b x + \log_b y \text{ for all } b > 1, x > 0 \text{ and } y > 0$$

$$= \frac{\lg 2 + 2\lg x}{\lg e} \quad \text{for all } x \geq 1 \quad \text{since } \log_b x^r = r \log_b x \text{ for all } b > 1, x > 0 \text{ and } r \in \mathbb{R}$$

$$\leq \frac{\lg x + 2\lg x}{\lg e} \quad \text{for all } x \geq 2 \quad \text{since } \lg 2 \leq \lg x \text{ for all } x \geq 2$$

$$= \frac{3\lg x}{\lg e} \quad \text{for all } x \geq 2$$

$$= c_2 |\lg x| \quad \text{for all } x \geq N, \text{ where } c_2 = \frac{3}{\lg e} \text{ and } N = 2$$

Thus, the upper bound $\ln(x^2 + 1) = O(\lg x)$ holds.

Next, we prove a lower bound.

$$\ln(x^2 + 1) \quad \geq \ln x^2 \quad \text{for all } x \geq 2 \quad \text{since } \ln x \text{ is a monotonically increasing function}$$

$$= 2\ln x \quad \text{for all } x \geq 2 \quad \text{since } \log_b x^r = r \log_b x \text{ for all } b > 1, x > 0 \text{ and } r \in \mathbb{R}$$

$$= \frac{2\lg x}{\lg e} \quad \text{for all } x \geq 2 \quad \text{by the base conversion}$$

$$= c_1 |\lg x| \quad \text{for all } x \geq N, \text{ where } c_1 = \frac{2}{\lg e} \text{ and } N = 2$$

Thus, the lower bound $\ln(x^2 + 1) = \Omega(\lg x)$ holds.

Therefore, the tight bound $\ln(x^2 + 1) = \Theta(\lg x)$ holds. \qed
Ex.5 Prove or disprove \( \sum_{i=1}^{n} \frac{1}{2(i+1)} = \Theta(n \lg n) \).

Theorem 5. \( \sum_{i=1}^{n} \frac{1}{2(i+1)} = \Theta(n \lg n) \)

Proof: We first simplify the summation.

\[
\sum_{i=1}^{n} \frac{1}{2(i+1)} = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i+1} = \frac{1}{2} \sum_{j=2}^{n+1} \frac{1}{j}
\]

Next, we prove the following lemma.

Lemma 1. \( \sum_{j=2}^{n} \frac{1}{j} = \Theta(\lg n) \)

Proof for Lemma 1: We first prove an upper bound.

\[
\sum_{j=2}^{n} \frac{1}{j} \leq \int_{1}^{n} \frac{1}{x} dx = \ln n - \ln 1 = \ln n \quad \text{since} \quad \ln 1 = 0
\]
\[
= \frac{\lg n}{\lg e} \quad \text{by the base conversion}
\]
\[
= c_2 |\lg n| \quad \text{for all} \quad n \geq N, \quad \text{where} \quad c_2 = \frac{1}{\lg e} \quad \text{and} \quad N = 2
\]

Next, we prove a lower bound.

\[
\sum_{j=2}^{n} \frac{1}{j} \geq \int_{2}^{n} \frac{1}{x} dx = \ln n - \ln 2 > \ln n - 1 \quad \text{since} \quad \ln 2 < 1
\]
\[
= \frac{\lg n}{\lg e} - 1 \quad \text{by the base conversion}
\]
\[
> \frac{\lg n}{2} - 1 \quad \text{since} \quad 1 < \lg e < 2
\]
\[
= \frac{\lg n}{4} + (\frac{\lg n}{4} - 1)
\]
\[
\geq \frac{\lg n}{4} \quad \text{for all} \quad n \geq 16 \quad \text{since} \quad \lg n \geq 4 \quad \text{for all} \quad n \geq 16
\]
\[ c_1 |\lg n| \quad \text{for all } n \geq N, \text{ where } c_1 = \frac{1}{4} \text{ and } N = 16 \]

Therefore, the tight bound holds.

Let \( c_1 = \frac{1}{4}, c_2 = \frac{2}{\lg e} \) and \( N = 16 \). By Lemma 1, the following holds for all \( n \geq N \).

\[
\begin{align*}
    c_1 |\lg n| &= \frac{1}{4} \lg n \leq \frac{1}{2} \sum_{j=2}^{n} \frac{1}{j} < \frac{1}{2} \sum_{j=2}^{n+1} \frac{1}{j} = \frac{1}{2} \sum_{j=2}^{n} \frac{1}{j} + \frac{1}{n+1} \leq \frac{1}{\lg e} |\lg n| + \frac{1}{n+1} < \frac{1}{\lg e} |\lg n| + 1 < \\
    \frac{1}{\lg e} |\lg n| + \frac{1}{\lg e} |\lg n| &= \frac{2}{\lg e} |\lg n| = c_2 |\lg n| \quad \text{since } \frac{|\lg n|}{\lg e} > 1 \text{ for all } n \geq N.
\end{align*}
\]

Thus, the tight bound holds. \( \square \)

References


